Fuzzy dynamic programming approach to hybrid multiobjective multistage decision-making problems

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Abstract

The purpose of this paper is to develop a new fuzzy dynamic programming approach for solving hybrid multiobjective multistage decision-making problems. We first present a methodology of fuzzy evaluation and fuzzy optimization for hybrid multiobjective systems, in which the qualitative and quantitative objectives are synthetically considered. The qualitative objectives are evaluated by decision-makers with linguistic variables and the quantitative objectives are converted into proper dimensionless indices. After getting the marginal evaluations for each objective, a new aggregation method based on the principle of fuzzy pattern recognition is developed to get a global evaluation for all objectives. With the global evaluation obtained, a fuzzy optimization process is performed. Then we present a dynamic optimization algorithm by incorporating the fuzzy optimization process with the conventional dynamic programming technique to solve hybrid multiobjective multistage decision-making problems. A characteristic feature of the approach proposed is that various objectives are synthetically considered by the fuzzy systematic technique instead of the frequently employed weighted average method. Finally, an illustrative example is also given to clarify the developed approach and to demonstrate its effectiveness. © 2001 Published by Elsevier Science B.V. All rights reserved.

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1. Introduction

Dynamic Programming (DP) is a powerful optimization apparatus for dealing with a large spectrum of complex problems involving sequential or multistage decision-making in many areas, e.g., control theory, pattern recognition, operations research, systems analysis, etc. Such problems occur and are relevant in virtually all human activities. There are many imprecise and uncertain factors due to man’s inherent subjectivity and vagueness in the articulation of their opinions. For obvious reasons, the analysis of multistage decision-making problems by conventional DP is rather difficult under fuzzy environments. Assuming that Zadeh’s fuzzy sets theory was an appropriate
way to deal with uncertainties and imprecision in real-world problems, DP was one of the earliest fundamental methodologies to which fuzzy sets theory was applied [4], leading to what might be called fuzzy dynamic programming (FDP). FDP has received wide attention in many research and application fields during the last ten years. Numerous contributions to FDP of both a foundational and an application character have appeared in the literature (cf. [2, 4, 8–13, 15–17, 22]). Excellent reviews of FDP appear in the literature [16, 17].

Recently, multiobjective dynamic programming (MODP), which relies heavily on the conventional DP technique, is developed as a technique for solving problems that involve various objectives (which are often conflicting) that possess the DP characteristics (refer [1, 5, 13, 19, 21]). Most methods proposed in the literature have a common character, i.e. they only deal with quantitative objectives. The essentials of the approach proposed in the literature are converting the multiobjective problem into a single-objective problem in some way, both crisp or fuzzy modeling, and then solving it by the conventional dynamic programming technique.

The purpose of this paper is to present a new effective and efficient fuzzy dynamic programming approach to hybrid multiobjective multistage decision-making problems with qualitative and quantitative objectives. Our approach is based fundamentally on the technique of fuzzy synthetic evaluation and fuzzy optimization for a multiobjective system and on the conventional principles of dynamic programming. The feature of our approach is that the fuzzy multiobjective optimization process is performed dynamically at each stage instead of converting the multiobjective problem into a single-objective problem.

The paper is structured as follows: In Section 2, the hybrid multiobjective multistage decision-making problem is formulated. Section 3 concerns the fuzzy synthetic evaluation and fuzzy optimization model for a hybrid multiobjective system. Section 4 designs an efficient and effective algorithm of fuzzy dynamic programming for solving hybrid multiobjective multistage decision-making problems. Section 5 gives a numerical example to demonstrate the effectiveness of the proposed approach and Section 6 concludes the paper.

2. Hybrid multiobjective multistage decision-making problems

In the general setting assumed here, we have a deterministic system whose dynamics are described by a state transition equation

$$s_{t+1} = f(s_t, x_t) \quad (t = 1, 2, \ldots, T), \tag{2.1}$$

where $s_t \in S_t$ is a (crisp) state variable taking its value in the set $S_t$ of values permitted at stage (time) $t$, and $x_t \in X_t$ is a (crisp) decision variable taking its value in the set $X_t$ of possible decisions at stage $t$; $S_t = \{a^{(1)}_t, a^{(2)}_t, \ldots, a^{(m)}_t\}$ and $X_t = \{d^{(1)}_t, d^{(2)}_t, \ldots, d^{(n)}_t\}$ ($t = 1, 2, \ldots, T$) are assumed to be finite throughout this paper.

The performance effects of the multistage decision-making process are evaluated by $M$ hybrid objectives $O = \{O_1, O_2, \ldots, O_M\}$, where some of the objectives are qualitative and others are quantitative. We assume that each objective $O_j$ ($j = 1, 2, \ldots, M$) is decomposable, that is

$$y_j(x) = \bigoplus_{t=1}^{T} y_j^{(t)}(x_t), \tag{2.2}$$

where $x = (x_1, x_2, \ldots, x_T); y_j^{(t)}(x_t)$ represents the performance result achieved by objective $O_j$ when decision $x_t$ is made at stage $t$ ($t = 1, 2, \ldots, T$) and $y_j(x) = y_j(x_1, x_2, \ldots, x_T)$ represents the overall performance effect achieved by $O_j$ when the decision sequence $\{x_1, x_2, \ldots, x_T\}$ is made during the multi-stage decision-making process. For simplicity, we present the performance results $y_j^{(t)}(x_t)$ in the cardinal form for the quantitative objectives and in the fuzzy linguistic variable form for the qualitative objectives, where the fuzzy linguistic variable values are in a finite set of natural evaluation words or phrase, such as $\{\text{Extremely Good}, \text{Very good}, \text{Good}, \text{Fair}, \text{Poor}, \text{Very Poor}, \text{Extremely Poor}\}$. Without loss of generality, we also suppose that the quantitative objectives are expected to maximize the total sum of their performance results achieved at each stage and the qualitative objectives are expected to optimize the whole synthetic effect of their performance results achieved at each stage. Thus, a hybrid multiobjective multistage (HMOMS) decision-making problem can be described as follows:
HMOMS: Given the initial state \( s_1 \in S_1 \), we seek an optimal sequence of decisions \( \{x_1^*, x_2^*, \ldots, x_T^*\} \) such that the hybrid objectives achieve their optimal performance effects simultaneously during the multistage decision-making process. That is

\[
(H)-\max_{x_t \in X_t} \{[y_1(x), y_2(x), \ldots, y_M(x)]\}
\]

where \((H)-\max\) represents the process of optimizing whole performance effects of the hybrid objectives.

It is worth noticing that the hybrid objectives often consist of a set of conflicting objectives that their optimal values cannot be achieved simultaneously. Instead of trying to find an optimal decision sequence \( \{x_1^*, x_2^*, \ldots, x_T^*\} \) such that every objective is optimal (usually this is impossible), we try to find a decision sequence \( \{x_1^*, x_2^*, \ldots, x_T^*\} \) at which every objective is satisfied. We also notice that the hybrid objectives may not only be classified into two-class-quantitative objectives and qualitative objectives, but may also have incommensurable units among the quantitative objectives, i.e., each quantitative objective may have a different unit of measurement. Hence, it cannot be converted directly into a single-objective multistage decision-making problem being solved by the conventional dynamic programming technique, just as most reports have done.

In this paper, we present an efficient fuzzy dynamic optimization approach to this kind of hybrid multi-objective multistage decision-making problem. The approach proposed here relies heavily on the technique of fuzzy synthetic evaluation and fuzzy optimization for multiobjective system and on the conventional principles of dynamic programming. We will provide a kind of fuzzy synthetic evaluation model and fuzzy optimization technique for hybrid multiobjective systems. Based on this fuzzy synthetic evaluation model and fuzzy optimization technique, a fuzzy dynamic optimization algorithm for solving hybrid multiobjective multistage decision-making problems is developed. In the following sections we will deal with these issues in detail.

3. Fuzzy evaluation and optimization model for hybrid multiobjective systems

We consider a finite set of \( M \) hybrid objectives \( O = \{O_1, O_2, \ldots, O_M\} \), and without loss of generality, they are all expected to be maximized to a certain performance index. We also assume that two prototypes of the performance index of hybrid objectives be given, name them “ideal index values” \( B^+ = (b_1^+, b_2^+, \ldots, b_M^+) \) and “anti-ideal index values” \( B^- = (b_1^-, b_2^-, \ldots, b_M^-) \). Here the ideal index values \( B^+ = (b_1^+, b_2^+, \ldots, b_M^+) \) and anti-ideal index values \( B^- = (b_1^-, b_2^-, \ldots, b_M^-) \) of objectives may be predetermined by historical data or by man’s subjective judgement.

As mentioned above, a hybrid multiobjective system often consists of a set of conflicting goals that their ideal values cannot be achieved simultaneously. Instead of trying to find an optimal solution \( x^* \) in the decision space \( D \) such that every objective is optimal (usually this is impossible), we try to find a solution \( x^* \in D \) to maximize the synthetic membership degree of optimum for all objectives, i.e. the grade of membership to which every objective is close to its “ideal” value. From this point of view, an optimal decision is the point in the decision space \( D \) at which the synthetic membership degree of optimum for all objectives is maximum. In this section, we present an approach to obtain the marginal evaluation of the membership degree of optimum for each objective and to aggregate these marginal evaluations into the global evaluation of the synthetic membership degree of optimum for whole objectives. Based on the global evaluation obtained, we can perform fuzzy optimization for hybrid multiobjective systems.

3.1. Marginal evaluation for single objective

For each particular objective \( O_j \in O \), by its marginal evaluation we mean a mapping \( \phi_j : D \rightarrow [0, 1] \) which tells us to what degree each decision \( x \in D \) makes the objective \( O_j \) close to its ideal value \( b_j^+ \). In other words, \( \phi_j(x) \in [0, 1] \) is the degree of compatibility between the ideal index value \( b_j^+ \) and the realized index value \( y_j(x) \) for the objective \( O_j \). According to the definition of a fuzzy set, \( \phi_j \) is just a fuzzy subset describing the fuzzy concept of “optimum for objective \( O_j \)” on decision space \( D \), and \( \phi_j(x) \in [0, 1] \) is the membership...
degree of \( x \in D \) to which \( x \) is compatible with the “optimal decision” considering the objective \( O_j \).

3.1.1. Quantitative objective

For each quantitative objective \( O_j \), let \( y_j(x) \) be the numerical value achieved by objective \( O_j \) for each decision \( x \in D \). As we want to maximize the numerical objective value, the ideal index values \( b_j^+ \) and antideal index values \( b_j^- \) can be derived from the numerical objective values simply by

\[
b_j^+ = \max_{x \in D} \{ y_j(x) \}, \quad b_j^- = \min_{x \in D} \{ y_j(x) \}
\]

\((j = 1, 2, \ldots, M)\). (3.1)

Since quantitative objectives may have incomensurable units, the value \( y_j(x) \) must be converted into proper dimensionless indexes for the purpose of comparison and trade-off analysis among quantitative objectives. That is, we should perform a marginal evaluation for each quantitative objective \( O_j \), which is just similar to the score function used by Fodor and Roubens [9].

In most practical applications, the following formula is employed to define the marginal evaluation mapping \( \phi_j : D \to [0, 1] \) for the quantitative objective \( O_j \):

\[
\phi_j(x) = \frac{y_j(x) - b_j^-}{b_j^+ - b_j^-} \quad (j = 1, 2, \ldots, M).
\]

3.1.2. Qualitative objective

The concept of a fuzzy linguistic variable is very useful in dealing with situations which are too complex or too ill-defined to be reasonably described in conventional quantitative expressions [23]. A fuzzy linguistic variable is a variable whose values are words or phrase in natural or artificial language. For example, “efficiency” is a linguistic variable if its values are linguistic rather than numerical. A fuzzy linguistic variable is often characterized by fuzzy number. The concept of fuzzy number is defined in the following [6, 7]:

**Definition 3.1** (Dubois and Prade [6, 7]). A fuzzy number \( \tilde{A} \) is a fuzzy set defined on the real line \( \mathbb{R} \) whose membership function \( f_\tilde{A}(x) \) has the following characteristics with \(-\infty \leq \alpha \leq \beta \leq \gamma \leq \delta \leq \infty\):

\[
f_\tilde{A}(x) = \begin{cases} 
  f_\tilde{A}^L(x), & \alpha \leq x \leq \beta, \\
  1, & \beta \leq x \leq \gamma, \\
  f_\tilde{A}^R(x), & \gamma \leq x \leq \delta, \\
  0, & \text{otherwise},
\end{cases}
\] (3.3)

where \( f_\tilde{A}^L : [\alpha, \beta] \to [0, 1] \), which is the left membership function of fuzzy number \( \tilde{A} \), is continuous and strictly increasing in \([\alpha, \beta]\), and \( f_\tilde{A}^R : [\gamma, \delta] \to [0, 1] \), which is the right membership function of fuzzy number \( \tilde{A} \), is continuous and strictly decreasing in \([\gamma, \delta]\).

We denote by \( \mathcal{F}(R) \) the set of all fuzzy numbers.

Many different membership functions can be defined for fuzzy numbers that possess the above-mentioned characteristics. A special example is the triangular function (Fig. 1).

**Definition 3.2.** A fuzzy number \( \tilde{A} \) is a triangular fuzzy number if its membership function \( f_\tilde{A} : \mathbb{R} \to [0, 1] \) is given by

\[
f_\tilde{A}(x) = \begin{cases} 
  (x - a)/(a - c), & c \leq x \leq a, \\
  (x - b)/(a - b), & a \leq x \leq b, \\
  0, & \text{otherwise},
\end{cases}
\]

where \( c \leq a \leq b \), and \( \tilde{A} \) is the triangular fuzzy number denoted by \( \tilde{A} = (c, a, b) \).

In this paper, let \( L = \{EG, VG, G, F, P, VP, EP\} \) denote the linguistic value set, where \( EG = \) Extremely Good, \( VG = \) Very Good, \( G = \) Good, \( F = \) Fair, \( P = \) Poor, \( VP = \) Very Poor and \( VG = \) Extremely Poor.
The linguistic values in $L$ are used to evaluate the performance of decisions versus qualitative objectives by the decision-maker(s) and they are characterized by triangular fuzzy numbers on $[0, 1]$ as follows (Fig. 2):

$$EG = (0.95, 1, 1):$$

$$f_{EG}(x) = \begin{cases} 
20x - 19, & 0.95 \leq x \leq 1, \\
0, & 0 \leq x < 0.95.
\end{cases}$$

$$VG = (0.7, 0.85, 1):$$

$$f_{VG}(x) = \begin{cases} 
\frac{1}{3}(20x - 14), & 0.7 \leq x \leq 0.85, \\
\frac{20}{3}(1 - x), & 0.85 < x \leq 1, \\
0, & \text{otherwise}.
\end{cases}$$

$$G = (0.55, 0.7, 0.85):$$

$$f_{G}(x) = \begin{cases} 
\frac{1}{3}(20x - 11), & 0.55 \leq x \leq 0.7, \\
\frac{1}{3}(17 - 20x), & 0.7 < x \leq 0.85, \\
0, & \text{otherwise}.
\end{cases}$$

$$F = (0.35, 0.5, 0.65):$$

$$f_{F}(x) = \begin{cases} 
\frac{1}{5}(20x - 7), & 0.35 \leq x \leq 0.5, \\
\frac{1}{5}(13 - 20x), & 0.5 < x \leq 0.65, \\
0, & \text{otherwise}.
\end{cases}$$

$$P = (0.15, 0.3, 0.45):$$

$$f_{P}(x) = \begin{cases} 
\frac{1}{3}(20x - 6), & 0.15 \leq x \leq 0.30, \\
\frac{1}{3}(9 - 20x), & 0.30 < x \leq 0.45, \\
0, & \text{otherwise}.
\end{cases}$$

$$VP = (0, 0.15, 0.3):$$

$$f_{VP}(x) = \begin{cases} 
\frac{20}{3}x, & 0 < x \leq 0.15, \\
\frac{1}{3}(6 - 20x), & 0.15 < x \leq 0.30, \\
0, & \text{otherwise}.
\end{cases}$$

$$EP = (0, 0, 0.05):$$

$$f_{EP}(x) = \begin{cases} 
1 - 20x, & 0 \leq x \leq 0.05, \\
0, & 0.05 < x \leq 1.
\end{cases}$$

In order to get a marginal evaluation for a qualitative objective, we introduce the concept of the total expected value of a fuzzy number, which identifies with the total integral value method proposed by Liou and Wang [18].

**Definition 3.3.** Let $\tilde{A}$ be a fuzzy number with left membership function $f_{\tilde{A}}^L$ and right membership function $f_{\tilde{A}}^R$. The total expected value of $\tilde{A}$ with an index of optimism $\alpha \in [0, 1]$ is defined as

$$E_\alpha(\tilde{A}) = \alpha E_R(\tilde{A}) + (1 - \alpha)E_L(\tilde{A}),$$

where $E_R(\tilde{A})$ and $E_L(\tilde{A})$ are the right and left expected values of $\tilde{A}$ defined as follows:

$$E_R(\tilde{A}) = \int_0^\beta x f_{\tilde{A}}^R(x) \, dx,$$

$$E_L(\tilde{A}) = \int_\gamma^0 x f_{\tilde{A}}^L(x) \, dx.$$ (3.4)

Equivalently, $E_R(\tilde{A})$ and $E_L(\tilde{A})$ can also be defined as

$$E_R(\tilde{A}) = \int_0^1 g_{\tilde{A}}^L(y) \, dy,$$ (3.5)

$$E_L(\tilde{A}) = \int_0^1 g_{\tilde{A}}^R(y) \, dy,$$

where $g_{\tilde{A}}^L(y)$ and $g_{\tilde{A}}^R(y)$ are the inverse functions of $f_{\tilde{A}}^L(x)$ and $f_{\tilde{A}}^R(x)$, respectively.

For a triangular fuzzy number $\tilde{A} = (c, a, b)$ and a level of optimism $\alpha \in [0, 1]$, it is easy to determine that $E_L(\tilde{A}) = \frac{1}{2}(a + c)$, $E_R(\tilde{A}) = \frac{1}{2}(a + b)$ and $E_\alpha(\tilde{A}) = \alpha E_R(\tilde{A}) + (1 - \alpha)E_L(\tilde{A}) = \frac{1}{2}(zb + a + (1 - \alpha)c)$.

The left expected value $E_L(\tilde{A})$ and right expected value $E_R(\tilde{A})$ of a triangular fuzzy number $\tilde{A}$ are of significance in geometry, i.e., they are the areas of trapezoids $OCPQ$ and $OBPQ$, respectively (refer Fig. 1). The parameter $\alpha \in [0, 1]$ reflects the decision-maker’s degree of optimism. A large $\alpha$ indicates a higher degree of optimism. More specifically, when $\alpha = 0$, $E_0(\tilde{A}) = E_L(\tilde{A})$ represents a pessimistic decision maker’s viewpoint. For an optimistic
For each qualitative objective $O_j$, let $y_j(x) \in L$ be the linguistic value result (characterized by a corresponding fuzzy number on $[0, 1]$) achieved by objective $O_j$ for each decision $x \in D$. Given the degree of optimism $\alpha \in [0, 1]$ (usually taking $\alpha = 0.5$), we can employ the $\alpha$-expected value to define the marginal evaluation for quantitative objective $O_j$. That is, if we define

$$\phi_j(x) = E_\alpha(y_j(x)),$$

then $\phi_j : D \rightarrow [0, 1]$ is the marginal evaluation mapping for quantitative objective $O_j$.

### 3.2. Global evaluation for multiple objectives

Having all the marginal evaluations $\phi_1(x), \phi_2(x), \ldots, \phi_M(x)$ for a given decision $x \in D$, the problem is then to determine a global evaluation of $x$ with respect to all the objectives. Thus, we want to define a mapping (score function) $\mu : D \rightarrow [0, 1]$ which tells us to what degree each decision $x \in D$ satisfies all of the objectives, or in other words, what is the global subjective evaluation value of decision $x$. Here $\mu : D \rightarrow [0, 1]$ is just a fuzzy subset describing the fuzzy concept of “optimum for all objectives” on decision space $D$, and $\mu(x) \in [0, 1]$ is the subjective membership degree of $x \in D$ to which $x$ is compatible with the “optimal decision” considering all objectives. An optimal decision is then the point in decision space $D$ at which the global subjective evaluation value (subjective membership degree) is maximum. Alternatively, the elements of decision space $D$ can be ranked according to their global subjection evaluation values when $D = \{x_1, x_2, \ldots, x_n\}$ is a finite set. That is, $\forall x_i, x_j \in D, x_j \gg x_j \iff \mu(x_i) \gg \mu(x_j)$.

The word subjective has been employed intentionally to stress that here, unlike in the marginal evaluation mechanism, the subjectivity of the decision-maker(s) must be taken into account, for different decision-makers with the same objectives will not reach the same decision. In this paper, we assumed that the subjectivity of the decision-maker(s) is reflected by their judgements on the relative importance of each objective. The relative importance is usually given by a set of weights which are normalized to sum to one, i.e.

$$w = (w_1, w_2, \ldots, w_M).$$

$$w_j \geq 0 \text{ and } \sum_{j=1}^M w_j = 1. \quad (3.7)$$

The weights can be assigned by the decision-maker(s) directly, or calculated using the eigenvector method in AHP, weighted least-squares method, logarithmic least-squares method, etc. (refer [14, 20]).

For given weights $w = (w_1, w_2, \ldots, w_M)$ and derived marginal evaluations $\phi_j(xA_w : [0, 1]^M \rightarrow [0, 1]$ such that $\forall x \in D$,

$$\mu(x) = (\phi_1(x), \phi_2(x), \ldots, \phi_M(x)). \quad (3.8)$$

In the literature on multiple-objective decision-making, the weighted arithmetic average is frequently employed as an aggregate operator. That is, the global evaluation mapping (score function) $\mu : D \rightarrow [0, 1]$ can be defined simply by

$$\mu(x) = \sum_{j=1}^M w_j \phi_j(x). \quad (3.9)$$

In the following contents in this section, we employ the fuzzy pattern recognition technique to derive an aggregating approach.

As pointed out above, we want to define a mapping $\mu : D \rightarrow [0, 1]$ such that $\mu(x)$ is the synthetic membership degree of the fuzzy concept “optimum” for decision $x$ with respect to all objectives. From the point of view of fuzzy pattern recognition, the decision space $D$ can be classified into several categories which often have vague boundaries, such as “good”, “fair”, “poor”, etc. These categories form a fuzzy partition of the decision space $D$. Each element of $D$ is matched with these categories optimally with a certain membership degree; here the optimal match will be characterized by a certain performance index. Especially, when the decision space $D$ is only classified into two categories of “optimum” and “not optimum”, the optimally matched membership degree of $x$ to the category “optimum” can be defined as the value of $\mu(x)$, which is just what we want. The details of this procedure proposed for defining the mapping $\mu : D \rightarrow [0, 1]$ may be described as follows:
For each decision \( x \in D \), its marginal evaluations with respect to the objective set \( O = \{ O_1, O_2, \ldots, O_M \} \) can be represented by an \( M \)-dimensional vector \( r(x) = (\phi_1(x), \ldots, \phi_M(x)) \). We assume that the decision space \( D \) is classified into \( c \) categories according to the marginal evaluations of its elements. The prototypes of the \( c \) categories are given also by their marginal evaluations, say \( p_k = (p_{k1}, p_{k2}, \ldots, p_{km}) \) \((k = 1, 2, \ldots, c)\). Suppose a decision \( x \in D \) is matched with these given \( c \) categories with membership degrees \( (\mu_1(x), \mu_2(x), \ldots, \mu_c(x)) \), respectively; here \( \mu_k(x) \in [0, 1] \) is the membership degree of \( x \) in the \( k \)th category satisfying
\[
\sum_{k=1}^{c} \mu_k(x) = 1 \quad (\forall x \in D). \tag{3.10}
\]

Considering the objectives are weighted with the eight vector \( w = (w_1, w_2, \ldots, w_M) \), we introduce a performance index \( J(\{\mu_k(x)\}) \) of this matching process in terms of the marginal evaluations by the following formula:
\[
J(\{\mu_k(x)\}) = \sum_{x \in D} \sum_{k=1}^{c} (\mu_k(x)\|w \cdot (r(x) - p_k)\|)^2, \tag{3.11}
\]
where \( \| \cdot \| \) is some inner product-induced norm in space \( R^M \) and \( \|w \cdot (r(x) - p_k)\| \) represents the weighted distance between the decision \( x \) and the prototype of \( k \)th category. Since \( \mu_k(x) \in [0, 1] \) is the membership degree of \( x \) in the \( k \)th category and (3.10) is satisfied, \( [\mu_k(x)||w \cdot (r(x) - p_k)||] \) can be viewed as the generalized weighted distance between the decision \( x \) and the prototype of the \( k \)th category. Hence the performance index given via (3.11) is of practical significance. Specifically, \( J(\{\mu_k(x)\}) \) is the total square sum of all generalized weighted distances between decisions and prototypes of categories. Clearly, the smaller the value of \( J(\{\mu_k(x)\}) \), the better the matching process. Therefore the optimal match problem is an optimization problem as follows:

\[
\text{minimize} \quad J(\{\mu_k(x)\}) = \sum_{x \in D} \sum_{k=1}^{c} (\mu_k(x)\|w \cdot (r(x) - p_k)\|)^2 \tag{3.12}
\]

subject to
\[
\sum_{k=1}^{c} \mu_k(x) = 1, \quad 0 \leq \mu_k(x) \leq 1. \tag{3.13}
\]

**Theorem 3.1.** Let the prototypes \( p_k = (p_{k1}, p_{k2}, \ldots, p_{km}) \) \((k = 1, 2, \ldots, c)\) and the vector \( w = (w_1, w_2, \ldots, w_m) \) be given. If \( \|w \cdot (r(x) - p_k)\| > 0 \) for all \( k = 1, 2, \ldots, c \), then \( (\mu_1(x), \mu_2(x), \ldots, \mu_c(x)) \) gives a minimum of (3.12) if and only if they are defined as
\[
\mu_k(x) = \frac{1}{\sum_{i=1}^{c} (\|w \cdot (r(x) - p_i)\|^2)} (k = 1, 2, \ldots, c). \tag{3.14}
\]

**Proof.** By the objective function (3.12) and the equality constraint condition in (3.13), we construct a Lagrange function \( L(\{\mu_k(x)\}) \) as
\[
L(\{\mu_k(x)\}; \lambda) = \sum_{k=1}^{c} (\mu_k(x)||w \cdot (r(x) - p_k)||)^2 + \lambda \left( 1 - \sum_{k=1}^{c} \mu_k(x) \right) \tag{3.15}
\]
and get the following derivatives:
\[
\frac{\partial L}{\partial \mu_k(x)} = 2\|w \cdot (r(x) - p_k)\|^2 \mu_k(x) - \lambda, \tag{3.16}
\]
\[
\frac{\partial L}{\partial \lambda} = 1 - \sum_{k=1}^{c} \mu_k(x), \tag{3.17}
\]
\[
\frac{\partial^2 L}{\partial \mu_k(x) \partial \mu_j(x)} = \begin{cases} 2\|w \cdot (r(x) - p_k)\|^2 & \text{if } k = j, \\ 0 & \text{otherwise}. \end{cases} \tag{3.18}
\]

Setting the derivatives (3.16) and (3.17) equal to zero, we get (3.14) immediately. From the derivative (3.18) we know that the Hessian matrix \( H = \text{diag}[2\|w \cdot (r(x) - p_1)\|^2, 2\|w \cdot (r(x) - p_2)\|^2, \ldots, 2\|w \cdot (r(x) - p_m)\|^2] \) is positive definite. Therefore \( J(\{\mu_k(x)\}) \) get its minimum at the point \( (\mu_1(x), \mu_2(x), \ldots, \mu_c(x)) \), where \( \mu_k(x) \) is given via (3.14). We have concluded the theorem. \( \square \)
Usually, the Euclidean distance in space $\mathbb{R}^M$ is used to take the place of the norm $\| \cdot \|$, i.e.
\[
\|w \cdot (r(x) - p_k)\| = \left( \sum_{j=1}^{M} [w_j(\phi_j(x) - p_{kj})]^2 \right)^{1/2},
\]  
(3.19)
where $\phi_j(x)$ is the marginal evaluation of $x$ with respect to objective $O_j$.

Especially, when $c = 2$, i.e., the decision space $D$ is only classified into two categories of “optimum” with prototypes $p_1 = (1, 1, \ldots, 1)$ and “not optimum” with prototype $p_2 = (0, 0, \ldots, 0)$ corresponding to the ideal objective value $B^+ = (b_1^+, b_2^+, \ldots, b_M^+)$ and anti-ideal objectives $B^- = (b_1^-, b_2^-, \ldots, b_M^-)$, then we can get the optimal matched membership degree, $\mu(x)$, of $x$ to the category “optimum” from Theorem 3.1 directly:
\[
\mu(x) = \frac{1}{\sum_{j=1}^{M} \frac{\|w \cdot (r(x) - p_1)\|^2}{\|w \cdot (r(x) - p_2)\|^2}}
= \frac{\sum_{j=1}^{M} [w_j(1-\phi_j(x))]^2}{\sum_{j=1}^{M} [w_j \phi_j(x)]^2}.
\]  
(3.20)

It is indicated by (3.20) that we have defined an $M$-place aggregation operator $\Phi_w : [0, 1]^M \rightarrow [0, 1]$ via
\[
(a_1, a_2, \ldots, a_M) = \frac{1}{\sum_{j=1}^{M} \frac{\|w \cdot (r(x) - a_j)\|^2}{\|w \cdot (r(x) - a_j)\|^2} + \sum_{j=1}^{M} [w_ja_j]^2} \sum_{j=1}^{M} [w_j(1-a_j)]^2
\]  
(3.21)
such that (3.8) holds $\forall x \in D$. Hence, the mapping $\mu : D \rightarrow [0, 1]$ defined via (3.20) is just the global evaluation mapping (score function) that we are looking for.

3.3. Fuzzy optimization and fuzzy optimal selection

After getting the global evaluation mapping $\mu : D \rightarrow [0, 1]$, we can define the fuzzy optimal decision $x^*$ as a point in $D$ at which the global evaluation is maximum, i.e.,
\[
\mu(x^*) = \max_{x \in D} \mu(x).
\]  
(3.22)

The corresponding objective values vector $Y(x^*) = (y_1(x^*), y_2(x^*), \ldots, y_M(x^*))$ is called the fuzzy optimum value of objectives vector $O = \{O_1, O_2, \ldots, O_M\}$ and the fuzzy optimal decision $x^*$ can be called a fuzzy optimal selection in $D$. The process of fuzzy optimization and fuzzy selection described above is denoted by
\[
Y(x^*) = F-Opt \max_{x \in D} \max_{j=1}^{M} (y_j(x), y_j(x), \ldots, y_j(x)) \max_{x \in D} \max_{j=1}^{M} (y_j(x), y_j(x), \ldots, y_j(x))
\]  
(3.23)

**Theorem 3.2.** If $w_j > 0$ for all $j = 1, 2, \ldots$, then the fuzzy optimum decision $x^*$ defined via (3.22) is a non-dominated decision (efficient decision or Pareto-optimal decision). That is, there is no other decision $x \in D$ such that
\[
y_j(x) \geq y_j(x^*)
\]  
(3.24)
holds for all $j = 1, 2, \ldots$, and
\[
y_j(x) > y_j(x^*)
\]  
(3.25)
holds for at least a $j_0 \in 1, 2, \ldots$. Hence it follows from (3.20) and $w_j > 0$ (for all) that
\[
\mu(x) = \frac{1}{\sum_{j=1}^{M} \frac{\|w \cdot (r(x) - a_j)\|^2}{\|w \cdot (r(x) - a_j)\|^2} + \sum_{j=1}^{M} [w_ja_j]^2} \sum_{j=1}^{M} [w_j(1-a_j)]^2
\]  
(3.26)
\[
> \frac{1}{\sum_{j=1}^{M} \frac{\|w \cdot (r(x) - a_j)\|^2}{\|w \cdot (r(x) - a_j)\|^2} + \sum_{j=1}^{M} [w_ja_j]^2} \sum_{j=1}^{M} [w_j(1-a_j)]^2
\]  
(3.27)
\[
= \mu(x^*).
\]  
(3.28)
This inequality is contrary to the fact that $x^*$ is the fuzzy optimum decision in the sense of (3.22). We end the proof. \(\square\)
4. Fuzzy dynamic programming approach to HMOMS decision-making problems

4.1. General principle of the fuzzy dynamic approach

Consider the HMOMS decision-making problem described in Section 2, i.e.,

\[(H) \max_{x_t \in X_t} \{[y_1(x), y_2(x), \ldots, y_M(x)] | s_{t+1} = f(s_t, x_t)\}, \tag{4.1}\]

where \((H)\)-max represents the process of optimizing hole performance effects of the hybrid objectives. As the objectives are hybrid, for the unity of performance, the performance effect of each objective can be measured by the sum of membership degrees of performance result to “optimum”. Hence, the following multistage dimensionless vector optimization problem is derived:

\[
\max_{x_t \in X_t} \{[R_1(x), R_2(x), \ldots, R_M(x)] | s_{t+1} = f(s_t, x_t)\}
\]

\[
= \max_{x_t \in X_t} \left\{ \left[ \sum_{i=1}^{T} r_1^{(i)}(x_t), \sum_{i=1}^{T} r_2^{(i)}(x_t), \ldots, \sum_{i=1}^{T} r_M^{(i)}(x_t) \right] | s_{t+1} = f(s_t, x_t) \right\}, \tag{4.2}\]

where \(r_j^{(i)}(x_t)\) is the membership degrees of performance result achieved by objective \(O_j\) to “optimum” when decision \(x_t\) is made at stage \(t\) and \(R_j(x) = \sum_{i=1}^{T} r_j^{(i)}(x_t)\) is the total sum of the membership degrees of performance result to “optimum” achieved by objective \(O_j\) when the decision sequence \(\{x_1, x_2, \ldots, x_T\}\) is made during the whole decision making process. In the following contents, the marginal evaluation mapping (3.2) and (3.3) derived in Section 3.1, corresponding to quantitative and qualitative objectives respectively, are employed to calculate \(r_j^{(i)}(x_t)\) \((j = 1, 2, \ldots, M; t = 1, 2, \ldots, T)\).

Instead of transforming the multistage dimensionless vector optimization problem (4.2) into a single-objective multistage decision problem solved by the conventional dynamic programming method, we try to optimize the vector dynamically by the fuzzy optimization technique, developed in Sections 3.2 and 3.3, incorporating the conventional dynamic optimization principle. That is

\[
F^{\text{Opt}} \max_{x_t \in X_t} \{[R_1(x), R_2(x), \ldots, R_M(x)] | s_{t+1} = f(s_t, x_t)\}
\]

where \(F^{\text{Opt}}\) represents the process of fuzzy optimization. If we define the stage vector \(F_t(x_t, s_t)\) at the \(t\)th stage by

\[
F_t(x_t, s_t) = \{[R_1^{(t)}(x_t), R_2^{(t)}(x_t), \ldots, R_M^{(t)}(x_t)] | s_{t+1} = f(s_t, x_t)\}, \tag{4.3}\]

and the fuzzy optimal vector \(F^*_t(s_t)\) at the \(t\)th stage by

\[
F^*_t(s_t) = F^{\text{Opt}} \max_{x_t \in X_t} \{[R_1^{(t)}(x_t), R_2^{(t)}(x_t), \ldots, R_M^{(t)}(x_t)] | s_{t+1} = f(s_t, x_t)\}.
\]

\[
= F^{\text{Opt}} \max_{x_t \in X_t} \left\{ \left[ \sum_{k=t}^{T} r_1^{(k)}(x_k), \sum_{k=t}^{T} r_2^{(k)}(x_k), \ldots, \sum_{k=t}^{T} r_M^{(k)}(x_k) \right] \right\}, \tag{4.5}\]

where \(R_j^{(t)}(x_t) = \sum_{k=t}^{T} r_j^{(k)}(x_k)\) and \(x_t = (x_t, x_{t+1}, \ldots, x_T)\). Then by the optimal principle of conventional dynamic programming, we have the backward recurrence relation appropriate to the definition (4.5):

\[
F^*_t(s_t) = F^{\text{Opt}} \max_{x_t \in X_t} \{[R_1^{(t)}(x_t), R_2^{(t)}(x_t), \ldots, R_M^{(t)}(x_t)]
\]

\[
+ F^*_{t+1}(s_{t+1}) \}
\]

\[
= F^{\text{Opt}} \max_{x_t \in X_t} \{[R_1^{(t)}(x_t), R_2^{(t)}(x_t), \ldots, R_M^{(t)}(x_t)]
\]

\[
+ F^*_{t+1}(f(s_t, x_t))\}
\]

\[
t = T, T-1, \ldots, 2, 1, \tag{4.6}\]

where \(x_t\) is the decision variable at stage \(t\); \(F^*_t(s_t)\) is computed and \(x^*_t = x^*_t(s_t)\) is determined for each \(s_t \in S_t\) \((t = 2)\). At \(t = 1\), \(F^*_1(s_1) = F^*_1(x^*_1)\) is the answer desired. The fuzzy optimal decision sequence \(\{x^*_1, x^*_2, \ldots, x^*_T\}\) can be traced by

\[
x^*_t = x^*(s^*_t), \quad t = 1, 2, \ldots, T \tag{4.7}\]
with the state transition equation

\[ s_i^t = s_1, \quad s_i^{t+1} = f(x_i^t, s_i^t), \quad t = 1, 2, \ldots, T - 1. \quad (4.8) \]

### 4.2. Performing process of the fuzzy dynamic approach

Using the backward recurrence relation (4.6), we detail the performing process of the fuzzy dynamic programming approach to the HMOMS decision-making problem as follows:

**Step 1:** For all \( j = 1, 2, \ldots, M \), given weight \( w_j \) and the performance results \( y_{ij}^{(t)}(x_i) \) of objective \( O_j \) for decision \( x_i \in X_i \) at each stage \( t \):

1. If \( O_j \) is a quantitative objective, then let

\[ r_j^{(t)}(x_i) = \frac{y_{ij}^{(t)}(x_i) - y_j^-}{y_j^+ - y_j^-}, \]

where

\[ y_j^+ = \max_{x_i \in X_i} y_{ij}^{(t)}(x_i), \quad y_j^- = \min_{x_i \in X_i} y_{ij}^{(t)}(x_i). \]

2. If \( O_j \) is a qualitative objective, then let

\[ r_j^{(t)}(x_i) = E_x(y_{ij}^{(t)}(x_i)), \]

where \( E_x(y_{ij}^{(t)}(x_i)) \) is the expected value of \( y_{ij}^{(t)}(x_i) \in L = \{EG, VG, G, F, P, VP, EP\} \) as described in Section 3.1.

**Step 2:** Set \( t := T \) and \( R_j^{(T+1)*}(s_{T+1}) := 0 \) for \( j = 1, 2, \ldots, M \).

**Step 3:** For each \( x_i \in X_i, s_t \in S_t \) and \( j = 1, 2, \ldots, M \), let

\[ R_j^{(t)}(s_t, x_t) = r_j(x_t) + R_j^{(t+1)*}(f(x_t, s_t)) \]

and set

\[ a_j := R_j^{(t)}(s_t, x_t). \]

**Step 4:** Apply fuzzy global evaluation (3.18) to \((a_1, a_2, \ldots, a_M)\) to get

\[ \mu_j(s_t, x_t) = \frac{1}{1 + \frac{\sum_{j=1}^M (w_j(1 - a_j))^2}{\sum_{j=1}^M (w_j a_j)^2}}. \]

and to determine \( x_j^*(s_t) \) by

\[ \mu_j(s_t, x_j^*(s_t)) = \max_{x_i \in X_i} \mu_j(s_t, x_i). \]

**Step 5:** Let \( R_j^{(t)}(x_t) = R_j^{(t)}(s_t, x_j^*(s_t)) \).

**Step 6:** If \( t > 1 \), then set \( t := t - 1 \) and turn to Step 3; If \( t = 1 \), then go to the next step.

**Step 7:** Trace backward to get the fuzzy optimal decision sequence

\[ x_j^* = x_j^*(s_t^*) \quad t = 1, 2, \ldots, T \]

with the state transition equation

\[ s_t^* = s_1, \quad s_{t+1}^* = f(x_t^*, s_t^*) \quad t = 1, 2, \ldots, T - 1. \]

**Step 8:** Stop.

**Theorem 4.1.** Let \( w_j \ (j = 1, 2, \ldots, M) \) be the weights of the \( M \) hybrid objectives. If \( w_j > 0 \) for all \( j = 1, 2, \ldots, M \), then the fuzzy-optimal decision sequence \( \{x_1^*, x_2^*, \ldots, x_T^*\} \) obtained above is a non-dominated (efficient or Pareto-optimal) decision. That is, there is no other decision sequence \( \{x_1, x_2, \ldots, x_T\} \) satisfying \( x_i \in X_i \) and \( s_{t+1} = f(x_t, s_t)(t = 1, 2, \ldots, T) \) such that

\[ R_j(x) \geq R_j(x^*) \]

holds for all \( j = 1, 2, \ldots, M \) and

\[ R_{j_0}(x) > R_{j_0}(x^*) \]

holds for at least a \( j_0 \in \{1, 2, \ldots, M\} \), where \( R_j(x) = \sum_{t=1}^T R_j^{(t)}(x_t) \) is the total sum of the membership degrees of performance result to "optimum" achieved by objective \( O_j \) when the decision sequence \( \{x_1, x_2, \ldots, x_T\} \).

<table>
<thead>
<tr>
<th>Number of workers</th>
<th>Jobs (efficiency, cost)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( J_1 )</td>
</tr>
<tr>
<td>1</td>
<td>(P, 60)</td>
</tr>
<tr>
<td>2</td>
<td>(F, 50)</td>
</tr>
<tr>
<td>3</td>
<td>(G, 40)</td>
</tr>
<tr>
<td>4</td>
<td>(VG, 40)</td>
</tr>
<tr>
<td>5</td>
<td>(VG, 45)</td>
</tr>
</tbody>
</table>
Table 2
The resulting fuzzy dynamic optimization calculations for $T = 4$

<table>
<thead>
<tr>
<th>State value $S_4$</th>
<th>Optimal decision $x_4^*$</th>
<th>Decision effect $(y_1^{(4)}(x^<em>), y_2^{(4)}(x^</em>))$</th>
<th>Stage sum of optimum degree (efficiency, cost)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>(EP, 130)</td>
<td>(0.0000, 0.0000)</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>(VP, 115)</td>
<td>(0.1500, 0.1364)</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>(P, 100)</td>
<td>(0.3000, 0.2727)</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>(F, 100)</td>
<td>(0.5000, 0.2727)</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>(G, 90)</td>
<td>(0.7000, 0.3636)</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>(G, 80)</td>
<td>(0.7000, 0.4545)</td>
</tr>
</tbody>
</table>

Table 3
The resulting fuzzy dynamic optimization calculations for $T = 3$

<table>
<thead>
<tr>
<th>State value $S_3$</th>
<th>Optimal decision $x_3^*$</th>
<th>Decision effect $(y_1^{(3)}(x^<em>), y_2^{(3)}(x^</em>))$</th>
<th>Stage sum of optimum degree (efficiency, cost)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>(EP, 85)</td>
<td>(0.0000, 0.4091)</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>(F, 60)</td>
<td>(0.5000, 0.6364)</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>(F, 60)</td>
<td>(0.6500, 0.7727)</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>(F, 60)</td>
<td>(0.8000, 0.9091)</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>(F, 60)</td>
<td>(1.0000, 0.9091)</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>(F, 60)</td>
<td>(1.2000, 1.0000)</td>
</tr>
</tbody>
</table>

Table 4
The resulting fuzzy dynamic optimization calculations for $T = 2$

<table>
<thead>
<tr>
<th>State value $S_2$</th>
<th>Optimal decision $x_2^*$</th>
<th>Decision effect $(y_1^{(2)}(x^<em>), y_2^{(2)}(x^</em>))$</th>
<th>Stage sum of optimum degree (efficiency, cost)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>(EP, 90)</td>
<td>(0.0000, 0.7727)</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>(EP, 90)</td>
<td>(0.5000, 1.0000)</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>(VP, 60)</td>
<td>(0.6500, 1.2727)</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>(F, 50)</td>
<td>(1.0000, 1.3636)</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>(F, 50)</td>
<td>(1.1500, 1.5000)</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>(F, 50)</td>
<td>(1.3000, 1.6364)</td>
</tr>
</tbody>
</table>

Table 5
The resulting fuzzy dynamic optimization calculations for $T = 1$

<table>
<thead>
<tr>
<th>State value $S_1$</th>
<th>Optimal decision $x_1^*$</th>
<th>Decision effect $(y_1^{(1)}(x^<em>), y_2^{(1)}(x^</em>))$</th>
<th>Stage sum of optimum degree (efficiency, cost)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>1</td>
<td>(P, 60)</td>
<td>(1.4500, 2.1364)</td>
</tr>
</tbody>
</table>
is made during the whole decision-making process ($j = 1, 2, \ldots, M; \ t = 1, 2, \ldots, T$).

**Proof.** Theorem 4.1 follows from Theorem 3.3 and the optimal principle of conventional dynamic programming directly.

Based on the performing process described above, a computational algorithm and computer programming for solving HMOMS decision-making problems by the fuzzy dynamic optimization approach proposed can be designed easily.

5. Illustrative example

Let us consider a resource allocation problem. Suppose there are 5 workers to be assigned to 4 jobs. Two objectives are considered; one is the quantitative objective “cost” which is expected to be minimized. The other is the qualitative objective “efficiency” which is evaluated with linguistic value set $L = \{EG, VG, G, F, P, VP, EP\}$, where $EG = \text{Extremely Good}$, $VG = \text{Very Good}$, $G = \text{Good}$, $F = \text{Fair}$, $P = \text{Poor}$, $VP = \text{Very Poor}$ and $EP = \text{Extremely Poor}$. Table 1 provides the expected values of efficiency and cost, respectively.

If we define the stage $t$ as the process of assigning the remaining $s_t$ workers among the jobs $J_t, J_{t+1}, \ldots, J_4$ ($t = 1, 2, 3, 4; \ s_1 = 5$), then this resource allocation problem is a hybrid bi-objective 4-stage decision problem.

In stage $t$, state variable $s_t \in \{0, 1, \ldots, 5\}$ represents the quantity of remaining workers to be assigned among the jobs $J_t, J_{t+1}, \ldots, J_4$, and decision variable $x_t \in \{0, 1, \ldots, s_t\}$ represents the quantity of workers to be assigned to the job $J_t$ from the $t$th job through the 4th activity. Obviously, the state transition equation is $s_{t+1} = s_t - x_t \ (t = 1, 2, 3, 4)$.

The resulting fuzzy dynamic optimization calculations for solving this HMOMS decision-making problem with weight vector $w = (w_1, w_2) = (0.5, 0.5)$ and optimism degree $\alpha = 0.5$ are given in Tables 2–5. Various weights assigned and the corresponding fuzzy optimal decision sequence $\{x_{1t}^*, x_{2t}^*, x_{3t}^*, x_{4t}^*\}$ and objective values are shown in Table 6. (In Table 6, the value of objective Efficiency is represented by total sum of its optimum degrees at each stage.)

Table 6.

<table>
<thead>
<tr>
<th>Weights $(w_1, w_2)$</th>
<th>Optimal decision sequence ${x_{1t}^<em>, x_{2t}^</em>, x_{3t}^<em>, x_{4t}^</em>}$</th>
<th>Objective values (efficiency, cost)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.0, 1.0)</td>
<td>0-2-1-2</td>
<td>(1.3000, 280)</td>
</tr>
<tr>
<td>(0.1, 0.9)</td>
<td>0-2-1-2</td>
<td>(1.3000, 280)</td>
</tr>
<tr>
<td>(0.2, 0.8)</td>
<td>0-2-1-2</td>
<td>(1.3000, 280)</td>
</tr>
<tr>
<td>(0.3, 0.7)</td>
<td>1-2-1-1</td>
<td>(1.4500, 285)</td>
</tr>
<tr>
<td>(0.4, 0.6)</td>
<td>1-2-1-1</td>
<td>(1.4500, 285)</td>
</tr>
<tr>
<td>(0.5, 0.5)</td>
<td>1-2-1-1</td>
<td>(1.4500, 285)</td>
</tr>
<tr>
<td>(0.6, 0.4)</td>
<td>2-2-1-0</td>
<td>(1.5000, 285)</td>
</tr>
<tr>
<td>(0.7, 0.3)</td>
<td>2-2-1-0</td>
<td>(1.5000, 290)</td>
</tr>
<tr>
<td>(0.8, 0.2)</td>
<td>2-2-1-0</td>
<td>(1.5000, 290)</td>
</tr>
<tr>
<td>(0.9, 0.1)</td>
<td>2-2-1-0</td>
<td>(1.5000, 290)</td>
</tr>
<tr>
<td>(1.0, 0.0)</td>
<td>2-2-1-0</td>
<td>(1.5000, 290)</td>
</tr>
</tbody>
</table>

Table 6 indicates that the results obtained through our approach are rather reasonable. That is, as the weight $w_1$ of objective Efficiency increases from 0.0 to 1.0 with step 0.1, the total sum of optimum degrees of the qualitative objective Efficiency increases while total sum of the quantitative objective Cost decreases monotonically. The results in Table 6 also shows that the fuzzy optimal decisions obtained by fuzzy dynamic optimization approach are all non-dominated (efficient or Pareto-optimal) decisions.

6. Conclusions

A new approach has been proposed in this paper using fuzzy dynamic programming for solving hybrid multiobjective multistage decision problems and its feasibility and effectiveness have been verified by an illustrative example. The concepts of fuzzy numbers and fuzzy linguistic variables are utilized to get marginal evaluations for qualitative objectives. A new aggregation method is developed to get the global evaluation for hybrid multiple-objective systems. The fuzzy dynamic programming algorithm presented in this paper aims to fuzzy-optimize the total sum of membership degrees of “optimum” achieved by hybrid objectives at each stage. Based on the fuzzy evaluation and fuzzy optimization technique proposed in this paper, other kinds of dynamic programming algorithms can also be designed to solve the HMOMS decision-making problems.
Acknowledgements

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References